Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and let $L : V \longrightarrow W$ be a linear transformation.

Suppose that $\dim(V) = n$ and $\dim(W) = m$ and let $\mathcal{B}$ be a basis of V.

We define $L(\mathcal{B}) = \{\vec{w} \in W \mid \vec{w} = L(\vec{v}) \text{ for some } \vec{v} \in \mathcal{B}\}$. That is, the set of all possible images of the vectors of $\mathcal{B}$ under L.

Which of the following statments are true? Select all true statements.

- [x] $L(\mathcal{B})$ is linearly independent
- [x] If $L(\mathcal{B})$ is a basis of W, then L is an isomorphism
- [x] $L(\mathcal{B})$ contains n distinct vectors
- ( ) None of the above

Option 1: $L(\mathcal{B})$ is linearly independent

Solution: This is **FALSE**. For example, let $L(vecv) = \vec{0}_W$ for all $\vec{v} \in V$. Then $L(\mathcal{B}) = \{\vec{0}_W\}$, which is linearly dependent.

Option 2: If $L(\mathcal{B})$ is a basis of W, then L is an isomorphism

Solution: This is **FALSE**. For example, let $V = \mathbb{R}^2$, $W = \mathbb{R}$ and let

$$L\begin{bmatrix} a \\ b \end{bmatrix} = a.$$

Letting $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis of V, we have that $L(\mathcal{B}) = \{1\}$. We know that $\{1\}$ is a basis of $\mathbb{R}$, but L is NOT an isomorphism as $L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$. Thus, the kerel of L is not trivial, so L is not injective.

Option 3: $L(\mathcal{B})$ contains n distinct vectors

Solution: This is **FALSE**. For example, let $V = \mathbb{R}^2 = W$ and let

$$L\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

Letting $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis of V, we have that $L(\mathcal{B}) = \{\vec{e}_1\}$. This contains only one element, while $n = 2$.

Char 1 Vector Space

V subspace of W. $\Rightarrow V \subseteq W$.

- Subspace　证明 3条　$V \rightarrow$ vector space

  Vector Space　证明 3条 + 8个 axioms.

  可以用subspace test 去证明 vector space

Span$(S)=V$　　$S=\{\vec{v}_1, \dots, \vec{v}_k\}$

- Spanning Set $\xrightarrow{\text{lin-indep}}$ basis

$\vec{x}\in V$. $\vec{x}=t_1\vec{v}_1+\cdots+t_k\vec{v}_k$

所有$\vec{x}\in V$都能用S里的index表示

唯一解为0

spanning set + lin-indep

$$\boxed{\dim(S)=\dim(V) \wedge \text{Span}(S)=V \Leftrightarrow S \text{ linearly-indep}}$$

Char 2 Linear Transformation

$L: V \to W$
$\quad \vec{v} \quad \vec{w}$

- linear - transformation (vector)

$$\boxed{\begin{aligned} L(\vec{x}+\vec{y}) &= L(\vec{x}) + L(\vec{y}) \\ L(c\vec{x}) &= cL(\vec{x}) \end{aligned}}$$

$[T]_B$ 咋求

$T: V \to W.$
$\quad B \quad C.$ 找出C所

- linear - mapping (matrix)

$$\boxed{\begin{aligned} & L(\vec{0}) = \vec{0} \\ & \mathrm{Range}(L) \in W \\ & \mathrm{Ker}(L) \in V \end{aligned}}$$

$\mathrm{Range}(L) = \{ L(\mathbf{X}) \in W : \mathbf{X} \in V \}$

$\mathrm{Ker}(L) = \{ \mathbf{X} \in V : L(\mathbf{X}) = \vec{0} \}$

- 永远存在 $A \leftrightarrow L \quad A[\vec{v}]_B = [L(\vec{v})]_C$
- ${}_C[L]_B = [[L(\vec{b}_1)]_C \cdots [L(\vec{b}_n)]_C]$

- Change of coordinates

$[\vec{v}]_C = {}_C I_B [\vec{v}]_B$

${}_C I_B \, {}_B I_C [\vec{v}]_C = [\vec{v}]_C \qquad {}_C[L]^{-1}_B = {}_B[L]_C$

- injective + surjective = isomorphism

one - to - one

injective	surjective
$\Leftrightarrow \mathrm{Ker}(L) = \{\vec{0}\}$	$\Leftrightarrow \mathrm{Range}(L) = W$
$\Leftrightarrow \mathrm{nullity}(L) = 0$	$\Leftrightarrow \mathrm{rank}(L) = \dim(W)$
$\otimes \dim(V) > \dim(W)$	$\otimes \dim(V) < \dim(W)$

- linear + inject + surject
- $\Leftrightarrow \boxed{\begin{aligned} & L(L^{-1}(\vec{w})) = \vec{w} \\ & L(L^{-1}(\vec{v})) = \vec{v} \end{aligned}}$

$L: V \to W \qquad \dim(V) = \mathrm{rank}(L) + \mathrm{nullity}(L)$

Char 3 Diagonalizable

一 $(\lambda, \vec{v})$

$A\vec{v} = \lambda \vec{v}$ eigenpair: $(\lambda, \vec{v})$

eigenspace $E_\lambda = \{\vec{v} \in \mathbb{F}^n : A\vec{v} = \lambda\vec{v}\} = \{\vec{v} : (A-\lambda I_n)\vec{v} = \vec{0}\} = Null(A-\lambda I_n)$

characteristic polynomial: $C_A(\lambda) = \det(A-\lambda I)$

$g(\lambda)$: $\dim(Null(A-\lambda I))$

$a(\lambda)$: $C_A(\lambda)$ 有几个根

$A_{n\times n}$:
$\boxed{1 \le g(\lambda) \le a(\lambda) \le n}$

求 $(\lambda, \vec{v})$ of $L: V \to V$.
- → 列出 ordered basis of V
- → $A = [L]_B$, 求 $C_A(\lambda)$
- → 解 $(A-\lambda I)\vec{v} = \vec{0}$.

判断 $L: V \to V$ 是否 diagonalizable
- → Let $A = [L]_B$
- → 求 $C_A(\lambda) \to a_\lambda$
- → $E_\lambda = Null(A-\lambda I) \to g_\lambda$.
- → 判断 $a_\lambda \overset{?}{=} g_\lambda$

一 diagonalization

A is diagonaliable

$\Leftrightarrow \exists P$: diag, $P^{-1}AP = D$ $(A$ similar to $D)$

$\Leftrightarrow \exists D = \{\vec{v}_1, \cdots, \vec{v}_n\}$ $\left(D = P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}\right)$

$\Leftarrow$ $A_{n\times n}$ 有 n 个不同的 λ.

$\Leftrightarrow \forall \lambda,\ g(\lambda) = a(\lambda)$ (Characterization of diagonalizability)

- $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$

Char 4 Inner Product

- def. inner product

conjugate symmetry 1. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$

linearity in 1st arg 2. $\langle \alpha\vec{v}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$ 3. $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

positive definiteness 4. $\langle \vec{v}, \vec{v} \rangle \geqslant 0$ $\langle \vec{v}, \vec{v} \rangle = 0 \Rightarrow \vec{v} = \vec{0}$

property: $\langle \vec{v}, \alpha\vec{w} \rangle = \bar{\alpha} \langle \vec{v}, \vec{w} \rangle$

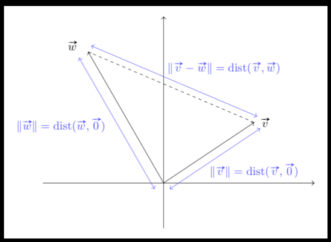
$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\| \Leftrightarrow \vec{v} = \alpha \vec{w}$ Cauchy-Schwarz inequality

standard inner product: $\langle \vec{v}, \vec{w} \rangle = \overline{\vec{w}^T} \vec{v}$

- norm: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

property: $\|\alpha\vec{v}\| = |\alpha| \|\vec{v}\|$ $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

$\text{dist}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$ normalization $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$



- orthogonal

$\vec{v} \perp \vec{w} \Rightarrow \langle \vec{v}, \vec{w} \rangle = 0$

$\Rightarrow \|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ Pythagorean Theorem

$(\leq)$

orthonormal

$\|\vec{v}\| = 1$

orthonormal basis: Gram-Schmidt Orthogonalization Procedure

$$\vec{w}_n = \vec{v}_n - \frac{\langle \vec{v}_n, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 - \cdots - \frac{\langle \vec{v}_n, \vec{w}_{n-1} \rangle}{\|\vec{w}_{n-1}\|^2} \vec{w}_{n-1}$$

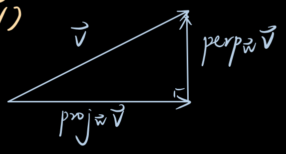
$\hat{w}_n$

orthogonal complement: $W^\perp = \{ \vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \}$

① 求 W 的 basis ② $\dim(V) = \dim(W) + \dim(W^\perp)$ ③ 根据定义求 $W^\perp$

proj: $\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}$ $\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{w}_k}(\vec{v})$

perp: $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}$ $\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v})$



- least square sol. (to $A\vec{x}=\vec{b}$) $\vec{s}$

$\vec{s}$ 是 $A\vec{x}=\text{proj}_{Col(A)}\vec{b}$ 的解.

给点求 best fit line.

① 定义 best fit line ep. $y=a+bx+cx^2$ $\vec{s}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $X=\begin{array}{ccc} x^0 & x^1 & x^2 \end{array}\left[\,(\ )\ (\ )\ (\ )\,\right]$

② $\vec{s}=(X^TX)^{-1}X^T\vec{y}$.

- Quadratic form $Q(\vec{u})=\vec{u}^TA\vec{u}$

positive definite	$Q(\vec{u})$	λ	>0
positive semi-definite	$Q(\vec{u})$	λ	≥ 0
negative definite	$Q(\vec{u})$	λ	<0
negative semi-definite	$Q(\vec{u})$	λ	≤ 0
indefinite	$Q(\vec{u})$	λ	$>/<0$

Chcr 5 Orthogonal Diagonalization

- adjoint $\qquad A^* = \overline{A}^T$

property: 1. $(A+B)^* = A^* + B^*$ $\qquad$ 2. $(AB)^* = B^* A^*$

3. $(A^*)^* = A$ $\qquad$ 4. $(\alpha A)^* = \bar{\alpha} A^*$

5. $\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A^*\vec{w}\rangle$ fundamental property of adjoint

$\boxed{\begin{array}{l} \cdot\ \langle U\vec{v}, U\vec{u}\rangle = \langle \vec{v}, \vec{u}\rangle \\ \cdot\ \|U\vec{v}\| = \|\vec{v}\| \end{array}}$

$$
\begin{tikzcd}
\boxed{\begin{array}{l}\text{unitary}\\ U^*U = UU^* = I\end{array}} \arrow[r, "\mathbb{R}"] \arrow[d, leftrightarrow, "+\operatorname{diag}(U^*AU)"] & \boxed{\begin{array}{l}\text{orthogonal}\\ Q^TQ = QQ^T = I\end{array}} \arrow[dd, leftrightarrow, "+\operatorname{diag}(Q^TAQ)."] \\
\boxed{\begin{array}{l}\text{normal}\\ A^*A = AA^*\end{array}} \arrow[d, "+\lambda\in\mathbb{R}"] & \\
\boxed{\begin{array}{l}\text{Hermitian}\\ A = A^*\end{array}} \arrow[r, "\mathbb{R}"] & \boxed{\begin{array}{l}\text{symmetric}\\ A = A^T\end{array}}
\end{tikzcd}
$$

property: 1. $\|A\vec{x}\| = \|A^*\vec{x}\|$

2. $A\vec{v} = \lambda\vec{v} \Rightarrow A^*\vec{v} = \bar{\lambda}\vec{v}$.

3. eig-vec $\vec{x}, \vec{y}$ 对应不同的 $\lambda \Rightarrow \vec{x} \perp \vec{y}$.

$D = PAP^{-1}$
P: eigen vector 组成

- Schur's Triangularization Theorem. $\quad U^*AU = T = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\quad$ (U is real and orthogonal)

- Cayley-Hamilton Theorem. $\quad C_A(A) = 0_{n\times n}$.

Gram matrix of $\langle\ ,\ \rangle$ with respect to $\underset{\{\vec{g}_1, \cdots, \vec{g}_n\}}{\underline{B}}$. $\qquad A = \begin{bmatrix} \langle \vec{g}_1, \vec{g}_1\rangle & \cdots & \vdots \\ \vdots & \ddots & \\ \cdots & & \langle \vec{g}_n, \vec{g}_n\rangle \end{bmatrix}$

unitarily diagonalize A: 1. 求 λ. $\vec{v} \to \hat{v}$ 2. 根据GS.求 $W^\perp$ 3. $D = U^*AU$

Char 6 Singular Value decomposition

相当于 $m \times n$ 的 $\lambda, \vec{v}$

$\vec{u}_1 \rightarrow \frac{1}{\sigma} A\vec{v}$
$\vec{u}_2$
$\vec{u}_3 \rightarrow$ G-S theo

singular value : $\sigma_i = \sqrt{\lambda_i}$ $\qquad$ $\vec{v}$: singular vector

singular value decomposition $\overset{m\times n}{A} = U \Sigma V^*$ $= \underset{3\times 3}{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix}^*}$

$Null(A^*A) = Null(A)$

$\sigma_1 \neq \sigma_2 \Rightarrow \sigma_1$ 与 σ_2 对应的 singular value orthogonal

$Col(A)^\perp = Null(A^*)$